

# NONCOMMUTATIVE GEOMETRY IN THE FRAMEWORK OF DIFFERENTIAL GRADED CATEGORIES

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## 1. INTRODUCTION

This expository note is meant to be a brief introduction to noncommutative geometry in a differential graded (DG) framework, *i.e.*, such categories playing the role of spaces. Keeping in mind the significance of certain correspondences in classical geometry, we are naturally led to include them as some sort of generalised morphisms of spaces. In this sense this geometry is *motivic*. However, although the construction of the category of noncommutative spaces will follow closely that of motives, the resulting category of noncommutative spaces will not even be additive (only semiadditive). Passing on to the homotopy categories of the DG categories (considered as spaces) one recovers most of the results known at the level of triangulated categories. However, in this setting one does not run into some unpleasant technical problems which one would otherwise have to deal with at the level of triangulated categories. The theory seems to be a blend between algebraic topology (or homotopical geometry) and algebraic geometry. There is a possibility of recasting many different models of noncommutative geometry in this general setting. We also include some pointers to some other areas of mathematics, which are well adapted to be seen in this context.

It must also be emphasized that this is noncommutative geometry ‘at a large scale’ (after Ginzburg [28]) and, therefore, there are some natural new phenomena which are not quite compatible with the classical picture. One such instance is the isomorphism between certain classical spaces, *e.g.*, an abelian variety and its dual, which need not be isomorphic as

classical spaces (varieties). For noncommutative geometry 'at a small scale', *i.e.*, viewed as a deformation of classical geometry one may look at, *e.g.*, [32],[43].

The outline of the construction presented here can be found in a recent preprint of Kontsevich [39]. In fact, this is a simplified version of the proposed one in *ibid*. Readers should also refer to the articles of J. Lurie and Toën-Vezzosi [45, 46, 66, 65], which seem to have developed a geometry based on a functor of points approach from simplicial algebras (equivalent to connective differential graded algebras) to simplicial sets. The form of geometry in their parlance is homotopical algebraic geometry (HAG) or derived algebraic geometry (DAG)<sup>1</sup>. The material presented here is mostly modelled on the ICM talk of Keller [37], though there are some minor deviations.

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## 2. NONCOMMUTATIVE GEOMETRY IN A DG FRAMEWORK

For a long time it was felt that the language of triangulated categories is deficient for many purposes in geometry. The language of DG categories seems to have resolved most of the technical and aesthetic problems. We first prepare the readers for the seemingly abstruse definition of the category of noncommutative spaces. We propose a theory over an algebraically closed field of characteristic zero and by *Lefschetz Principle* there is no harm in assuming our ground field  $k$  to be actually  $\mathbb{C}$ . This reduces a lot of technical difficulties. For brevity, we denote most of the functors by their underived notation, for instance,  $\otimes^{\mathbb{L}}$  is written simply as  $\otimes$ .

**2.1. Motivation.** The traditional way of doing geometry with the emphasis on spaces is deficient in many physical situations. Most notably, due to Heisenberg's Uncertainty Principle one is forced to consider polynomial algebras with noncommuting variables, like Weyl algebras. One has to do away with the notion of points of a space quite naturally. However, one has perfectly well-defined algebras, albeit noncommutative, with which one can work. One very successful approach from this point of view is that of Connes [15]. It has many applications and a large part of the classical (differential) geometry can be subsumed in this setting. One might also want to take a closer look at the key features of classical (algebraic) geometry and try to generalise them.

**From spaces to categories; from functions to sheaves:** It is quite common in mathematics to study an object via its representations (in an appropriate sense). It is neat to assemble all representations into a category and study it. In this manner from groups one is led to the study of Tannakian categories, from algebras to that of certain triangulated categories and so on. This process is roughly some sort of categorification.

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<sup>1</sup>The author would like to thank D. Ben-Zvi for providing quite convincing arguments to dispel any idea that HAG or DAG can subsume noncommutative geometry

We have already done away with the traditional notion of a space and its points. For the time being it is described by its functions. The topology of a space allows us to define functions locally and glue them (if possible to a global one). A better way of keeping track of such information is using the language of the sheaf of local sections or functions on a space. Every classical space comes hand in hand with its structure sheaf of admissible functions *e.g.*, continuous, smooth, holomorphic, algebraic, etc. according to the structure of the underlying space. The representations of the structure sheaf, which for us are nothing but quasicoherent sheaves, determine the space. In this manner one replaces the notion of a space by its category of quasicoherent sheaves, an idea that goes back to Gabriel, Grothendieck, Manin and Serre.

The category of quasicoherent sheaves is a Grothendieck category when the scheme is quasicompact and quasiseparated [62]. There are many approaches towards developing a theory by treating abelian categories (or some modifications thereof, like Grothendieck categories) as the category of quasicoherent sheaves on noncommutative spaces, *e.g.*, [1, 67, 55].

**Remark 2.1.** *There is another point of view inspired by the Geometric Langlands programme and the details can be found, for instance, in [25]. The guiding principle here is a generalisation of Grothendieck's faisceaux-fonctions correspondence. The faisceaux-fonctions correspondence appears naturally in the context of étale  $\ell$ -adic sheaves. Associated to any complex of étale  $\ell$ -adic sheaves  $\mathcal{K}^\bullet$  over a variety  $V$  defined over a finite field  $\mathbb{F}_q$  is a function  $f^{\mathcal{K}^\bullet} : V \rightarrow \mathbb{C}$  given by*

$$f^{\mathcal{K}^\bullet}(x) = \sum (-1)^i \text{Tr}(\text{Fr}_{\bar{x}} | H^i(\mathcal{K}^\bullet)_{\bar{x}}).$$

Here  $x \in V(\mathbb{F}_q)$  and  $\bar{x}$  denotes a geometric point of  $V$  over  $x$ . Of course, one has to fix an identification  $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ . According to Grothendieck all interesting functions appear in this manner and extrapolating this idea we regard constructible sheaves as the only source of interesting functions over  $\mathbb{C}$ .

The lack of Verdier Duality, which is a generalisation of Poincaré Duality and hence an important feature, makes the naïve category of constructible  $\ell$ -adic sheaves undesirable. Instead one works with the category of so-called perverse sheaves.<sup>2</sup> They are objects which live in a bigger derived category. Via a version of the Riemann-Hilbert correspondence over  $\mathbb{C}$  the category of perverse sheaves (of middle perversity) is equivalent to the category of regular holonomic  $\mathcal{D}$ -modules. More precisely, let  $X$  be a complex manifold,  $D_{rh}^b(\mathcal{D}_X)$  denote the bounded derived category of complexes of  $\mathcal{D}_X$ -modules with regular holonomic cohomologies and  $D_c^b(\mathbb{C}_X)$  denote the bounded derived category of sheaves of complex vector spaces with constructible cohomologies. Then Kashiwara proved in [33]  $\mathcal{R}\mathcal{H}\text{om}_{\mathcal{D}_X}(-, \mathcal{O}_X) : D_{rh}^b(\mathcal{D}_X) \xrightarrow{\sim} D_c^b(\mathbb{C}_X)^{\text{op}}$  is an equivalence of triangulated categories. Under this equivalence the standard t-structure on  $D_{rh}^b(\mathcal{D}_X)$ , whose heart is the abelian category of regular holonomic  $\mathcal{D}$ -modules on  $X$ , is mapped to the heart of the t-structure of middle perversity on  $D_c^b(\mathbb{C}_X)$ . The heart of this t-structure is the category of perverse sheaves (of middle perversity), which can be regarded as another generalisation of functions. As opposed to a quasicoherent sheaf, the model for a function in this setting is a  $\mathcal{D}$ -module, which is roughly a quasicoherent sheaf

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<sup>2</sup>It is known that the derived category of coherent sheaves also admits a dualising complex imitating Grothendieck–Serre duality in place of Verdier duality (see Proposition 1 [5]).

with a flat connection. A quasicoherent sheaf (resp. a  $\mathcal{D}$ -module) corresponds to a polynomial (resp. a constructible locally constant) function.

**The passage to derived categories:** In the category of smooth schemes any morphism  $f : X \rightarrow Y$  gives rise to two canonical functors on the category of sheaves, *viz.*, pull-back  $f^*$  and push-forward  $f_*$ . One should naturally expect any generalisation of classical geometry to allow such operations. We see that restricting to abelian categories is not enough as functors like push-forwards are not exact. The natural framework for such functors to exist is that of derived categories or abstract triangulated categories. Besides, if one chooses to work with perverse sheaves as substitutes for functions one has to view them as elements of an abelian category sitting inside a bigger derived category.

**Adding correspondences to morphisms:** Denoting by  $\mathbf{Var}$  the category of complex algebraic varieties,  $\mathbf{Top}$  that of *nice* topological spaces (here *nice* should imply all properties typical of the complex points of a complex algebraic variety) one has a tensor functor  $\mathbf{Var} \rightarrow \mathbf{Top}$  associating to a complex algebraic variety its underlying space with analytic topology. The tensor structure on the two categories is given by direct product. To a topological space in  $\mathbf{Top}$  one can associate its singular cochain complex which is also a tensor functor to  $D_{ab}$ , the category of complexes of finitely generated abelian groups whose cohomology is bounded. According to Beilinson and Vologodsky [4] the basic objective of the theory of motives is to fill in a commutative diagram

$$\begin{array}{ccc} \mathbf{Var} & \longrightarrow & \mathcal{D}_{\mathcal{M}} \\ \downarrow & & \downarrow \\ \mathbf{Top} & \longrightarrow & D_{ab} \end{array}$$

where  $\mathcal{D}_{\mathcal{M}}$  is the rigid tensor triangulated category of motives. The upper horizontal arrow should be faithful and defined purely geometrically and the right vertical arrow should respect the tensor structures. In order to construct the upper horizontal arrow one first needs to enrich  $\mathbf{Var}$  to include correspondences (modulo some equivalence relation). This endows  $\mathbf{Var}$  with an additive structure.

**Triangulated structure is not enough:** The goal is to construct a rigid tensor category of *motivic* noncommutative spaces which allows basic operations like pull-back, push-forward and finite correspondences (as morphisms). In the classical setting, we have a construction of  $\mathcal{D}_{\mathcal{M}}$  as a triangulated category due to Voevodsky (see *e.g.*, [26]). However, one would like to extract the *right* category of motives inside it (possibly as an abelian rigid tensor category). One basic operation is direct product, which endows  $\mathbf{Var}$  with the tensor structure. It should also survive in  $\mathcal{D}_{\mathcal{M}}$ . The tensor product of two triangulated categories unfortunately does not carry a natural triangulated structure. Also one runs into trouble in trying to define inner Hom's. This is where the framework of DG (differential graded) categories comes in handy.

**2.2. Overview of DG categories.** Before we are able to spell out the definition of the category of noncommutative spaces we need some preparation on DG categories, which will be quite concise. For details we refer the readers to *e.g.*, [23],[37],[63]. They can be defined

over  $k$ , where  $k$  is not necessarily a field. However, as mentioned before, we set  $k = \mathbb{C}$  and, unless otherwise stated, all our categories are assumed to be  $k$ -linear.

A category  $\mathcal{C}$  is called a DG category if for all  $X, Y \in \text{Obj}(\mathcal{C})$   $\text{Hom}(X, Y)$  has the structure of a complex of  $k$ -linear spaces (in other words, a DG vector space) and the composition maps are associative  $k$ -linear maps of DG vector spaces. In particular,  $\text{Hom}(X, X)$  is a DG algebra with a unit.

**Example 1.** Given any  $k$ -linear category  $\mathcal{M}$  it is possible to construct a DG category  $\mathcal{C}_{dg}(\mathcal{M})$  with complexes  $(M^\bullet, d_M)$  over  $\mathcal{M}$  as objects and setting  $\text{Hom}(M^\bullet, N^\bullet) = \bigoplus_n \text{Hom}(M^\bullet, N^\bullet)_n$ , where  $\text{Hom}(M^\bullet, N^\bullet)_n$  denotes the component of morphisms of degree  $n$ , i.e.,  $f_n : M^\bullet \rightarrow N^\bullet[n]$  and whose differential is the graded commutator

$$d(f) = d_M \circ f_n - (-1)^n f_n \circ d_N.$$

Let  $\mathbf{DGcat}$  stand for the category of all small DG categories. The morphisms in this category are *DG functors*, i.e.,  $F : \mathcal{C} \rightarrow \mathcal{C}'$  such that for all  $X, Y \in \text{Obj}(\mathcal{C})$

$$F(X, Y) : \text{Hom}(X, Y) \longrightarrow \text{Hom}(FX, FY)$$

is a morphism of DG vector spaces compatible with the compositions and the units.

**The tensor structure:** The tensor product of two DG categories  $\mathcal{C}$  and  $\mathcal{D}$  can be defined in the obvious manner, *viz.*, the objects of  $\mathcal{C} \otimes \mathcal{D}$  are written as  $X \otimes Y$ ,  $X \in \text{Obj}(\mathcal{C})$ ,  $Y \in \text{Obj}(\mathcal{D})$  and one sets

$$\text{Hom}_{\mathcal{C} \otimes \mathcal{D}}(X \otimes Y, X' \otimes Y') = \text{Hom}_{\mathcal{C}}(X, X') \otimes \text{Hom}_{\mathcal{D}}(Y, Y')$$

with natural compositions and units.

The category of DG functors  $\mathcal{H}om(\mathcal{C}, \mathcal{D})$  between two DG categories  $\mathcal{C}$  and  $\mathcal{D}$  with natural transformations as morphisms is once again a DG category. With respect to the above-mentioned tensor product  $\mathbf{DGcat}$  becomes a symmetric tensor category with an inner  $\mathcal{H}om$  functor given by

$$\text{Hom}(\mathcal{B} \otimes \mathcal{C}, \mathcal{D}) = \text{Hom}(\mathcal{B}, \mathcal{H}om(\mathcal{C}, \mathcal{D})).$$

However, in the category of noncommutative spaces (to be defined shortly), this notion of the inner Hom functor needs to be modified.

**The derived category of a DG category:** The standard reference for the construction is [34]. We recall some basic facts here. Let  $\mathcal{C}$  be a small DG category. A right DG  $\mathcal{C}$ -module is by definition a DG functor  $M : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}_{dg}(k)$ , where  $\mathcal{C}_{dg}(k)$  denotes the DG category of complexes of  $k$ -linear spaces. Note that the composition of morphisms in the opposite category is defined by the Koszul sign rule: the composition of  $f$  and  $g$  in  $\mathcal{C}^{\text{op}}$  is equal to the morphism  $(-1)^{|f||g|}gf$  in  $\mathcal{C}$ . Every object  $X$  of  $\mathcal{C}$  defines canonically what is called a free right module  $X^\wedge := \text{Hom}(-, X)$ . A morphism of DG modules  $f : L \rightarrow M$  is by definition a morphism (natural transform) of DG functors such that  $fX : LX \rightarrow MX$  is a morphism of complexes for all  $X \in \text{Obj}(\mathcal{C})$ . We call such an  $f$  a quasiisomorphism if  $fX$  is a quasiisomorphism for all  $X$ , i.e.,  $fX$  induces isomorphism on cohomologies.

**Definition 2.2.** *The derived category  $D(\mathcal{C})$  of  $\mathcal{C}$  is defined to be the localisation of the category of right DG  $\mathcal{C}$ -modules with respect to the class of quasiisomorphisms.*

**Remark 2.3.** *With the translation induced by the shift of complexes and triangles coming from short exact sequences of complexes,  $D(\mathcal{C})$  becomes a triangulated category. The Yoneda functor  $X \mapsto X^\wedge$  induces an embedding of  $H^0(\mathcal{C}) \rightarrow D(\mathcal{C})$ . Here  $H^0(\mathcal{C})$  stands for the zeroth cohomology category whose objects are the same as  $\mathcal{C}$  but the morphisms are replaced by the zeroth cohomology, i.e.,  $\text{Hom}_{H^0(\mathcal{C})}(X, Y) = H^0 \text{Hom}_{\mathcal{C}}(X, Y)$ . It is also called the homotopy DG category as it produces the homotopy category of complexes over any  $k$ -linear category  $\mathcal{M}$  when specialised to  $\mathcal{C}_{dg}(\mathcal{M})$ .*

**Definition 2.4.** *The triangulated subcategory of  $D\mathcal{C}$  generated by the free DG  $\mathcal{C}$ -modules  $X^\wedge$  under translations in both directions, extensions and passage to direct factors is called the perfect derived category and denoted by  $\text{per}(\mathcal{C})$ . A DG category  $\mathcal{C}$  is said to be pretriangulated<sup>3</sup> if the above-mentioned Yoneda functor induces an equivalence  $H^0(\mathcal{C}) \rightarrow \text{per}(\mathcal{C})$ .*

**Remark 2.5.** *A pretriangulated category does not have a triangulated structure. Rather it is a DG category, which is equivalent to the notion of an enhanced triangulated category in the sense of Bondal–Kapranov [11], whose homotopy category is Karoubian.*

**Definition 2.6.** *A DG functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called a Morita equivalence if it induces an equivalence  $F^* : D(\mathcal{D}) \rightarrow D(\mathcal{C})$ .*

**2.3. The category of noncommutative spaces.** The definition provided below is a culmination of the works of several people spanning over two decades including Bondal, Drinfeld, Keller, Kontsevich, Lurie, Orlov, Quillen and Toën, amongst others. This list of names is very far from complete and it only reflects the authors ignorance of the history behind this development.

**Definition 2.7.** *The category of noncommutative spaces  $\mathbf{NCS}$  is the localisation of  $\mathbf{DGcat}$  with respect to Morita equivalences.*

Thanks to Tabuada we know that  $\mathbf{DGcat}$  has a cofibrantly generated Quillen model category structure, where the weak equivalences are the Morita equivalences and the fibrant objects are pretriangulated DG categories. It seems that there was a slight inaccuracy in the proof of the above statement that appeared in [59], which has now been corrected in [58]. This enables us to conclude that each object of  $\mathbf{NCS}$  has a fibrant replacement, which is a pretriangulated DG category. The tensor product of  $\mathbf{DGcat}$  induces one on  $\mathbf{NCS}$  after replacing any object by its cofibrant model since the tensor product by a cofibrant DG module preserves weak equivalences. The category  $\mathbf{NCS}$  can be regarded as a enhancement of the category of all small idempotent complete triangulated categories.

**Remark 2.8.** *We have deliberately included correspondences in the category of noncommutative spaces. These spaces are somewhat motivic in nature and it is expected to be a feature of this geometry. We do not want to treat  $\mathbf{NCS}$  as a 2-category.*

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<sup>3</sup>Our definition of a pretriangulated category is slightly stronger than [37], in that, in our definition the homotopy category of such a category is automatically idempotent complete.

However, the inner  $\mathcal{H}om$  functor cannot be derived from  $\mathbf{DGcat}$ . Thanks to Toën [64] (also *cf.* [36]) one knows that there does exist an inner  $\mathcal{H}om$  functor given by

$$(1) \quad \mathcal{H}om(\mathcal{C}, \mathcal{D}) = \text{cat. of } A_\infty\text{-functors } \mathcal{C} \rightarrow \mathcal{D}$$

here  $\mathcal{D}$  needs to be a pretriangulated DG category which is no restriction since we know that in  $\mathbf{NCS}$  every object has a canonical pretriangulated replacement. The DG structure of  $\mathcal{D}$  endows  $\mathcal{H}om(\mathcal{C}, \mathcal{D})$  with a DG structure as well. We will not be able to discuss  $A_\infty$ -categories and  $A_\infty$ -functors here. Let us mention that a DG category is a special case of an  $A_\infty$ -category and we refer the readers to, *e.g.*, [36] for a highly readable survey of the same.

**Remark 2.9.** *The Hom sets in  $\mathbf{NCS}$  are commutative monoids and it is possible to talk about exact sequences in  $\mathbf{NCS}$  (see Definition 3.5 below).*

**Definition 2.10** (Kontsevich).

- A noncommutative space (DG category)  $\mathcal{C}$  is called smooth if the bimodule given by the DG bifunctor  $(X, Y) \mapsto \text{Hom}_{\mathcal{C}}(X, Y)$  is in  $\text{per}(\mathcal{C}^{\text{op}} \otimes \mathcal{C})$ .
- It is called smooth and proper if it is isomorphic in  $\mathbf{NCS}$  to a DG algebra whose homology is of finite total dimension.

**Remark 2.11.** *There is a notion of affinity in this context which just says that a variety is D-affine (or derived affine, *e.g.*, [6] for an analogous notion in the setting of  $\mathcal{D}$ -modules) if its triangulated category of quasicoherent sheaves is equivalent to the derived category of modules over some (possibly DG) algebra. A theorem of Bondal–Van den Bergh [10] (see also [56]) asserts that if  $X$  is a quasicompact and quasiseparated scheme, then  $D_{\text{QCoh}}(X)$  is equivalent to  $D(\Lambda)$  for a suitable DG algebra  $\Lambda$  with bounded cohomology. Note that in this theorem  $D_{\text{QCoh}}(X)$  denotes the honest derived category of complexes of  $\mathcal{O}_X$ -modules with quasicoherent cohomologies and  $D(\Lambda)$  likewise. As a consequence we deduce that in the DG setting every proper variety is D-affine.*

**Viewing classical geometry in this setting:** We define the *DG category of quasicoherent sheaves* on an honest scheme  $X$  as

$$\mathcal{C}_{dg}(X) := \mathcal{C}_{dg}(\text{QCoh}(X)) = \text{DG category of fibrant unbounded complexes over } \text{QCoh}(X),$$

which is how we view classical schemes in this framework. It is also known that  $H^0 \mathcal{C}_{dg}(X) \xrightarrow{\sim} D_{\text{QCoh}}(X)$ . As mentioned above there are reconstruction Theorems available from  $\text{QCoh}(X)$  (without any further assumption [27, 55]) and from  $D_{\text{QCoh}}(X)$  (only if the canonical or the anticanonical bundle is ample [9] or with the knowledge of the tensor and the triangulated structure [2]). They glaringly exclude abelian varieties or (weak) Calabi–Yau varieties, however, for abelian varieties we do have an understanding of the derived category and its autoequivalences [51].

**Remark 2.12.** *Those who prefer regular holonomic  $\mathcal{D}$ -modules as substitutes for functions can perform the above operation after replacing  $\text{QCoh}(X)$  by the category of regular holonomic  $\mathcal{D}$ -modules.*

Since we have enhanced the morphisms between our spaces by incorporating certain right perfect correspondences, we have also increased the chance of objects becoming isomorphic. Due to Mukai [50] we know that an abelian variety is derived equivalent to its dual precisely via a *correspondence-like* morphism, which is a Fourier–Mukai transform. Roughly, given any two smooth projective varieties  $X$  and  $Y$  and an object in  $\mathcal{E} \in D^b(X \times Y)$  one constructs an exact Fourier–Mukai transform (also sometimes called an integral transform)  $\Phi_{X \rightarrow Y}^\mathcal{E} : D^b(X) \longrightarrow D^b(Y)$  as follows:

$$\Phi_{X \rightarrow Y}^\mathcal{E}(-) = \pi_{Y*}(\mathcal{E} \otimes \pi_X^*(-)),$$

where  $\pi_X$  (resp.  $\phi_Y$ ) denotes the projection  $X \times Y \rightarrow X$  (resp.  $X \times Y \rightarrow Y$ ). Here all functors are assumed to be appropriately derived. The object  $\mathcal{E}$  is called the *kernel* of the Fourier–Mukai transform. In the case of the equivalence between an abelian variety  $A$  and its dual  $\hat{A}$  the kernel is given by the Poincaré sheaf  $\mathcal{P}$ . Given a divisorial correspondence in  $X \times Y$  one can consider the corresponding line bundle on  $X \times Y$  and use that as the kernel of a Fourier–Mukai transform. Conversely, given a kernel  $\mathcal{E} \in D^b(X \times Y)$  of a Fourier–Mukai transform one obtains a cycle (correspondence modulo an equivalence relation) in  $X \times Y$  by applying the Chern character to  $\mathcal{E}$ .

**2.4. DG categories up to quasiequivalences.** We gave a direct method of constructing the category  $\mathbf{NCS}$ . There is an intermediate notion which one might also want to consider. We call a DG functor  $F : \mathcal{C} \longrightarrow \mathcal{D}$  a *quasiequivalence* if the induced maps  $\mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(FX, FY)$  are quasiisomorphisms for all  $X, Y \in \mathrm{Obj}(\mathcal{C})$  and the induced functor  $H^0(F) : H^0(\mathcal{C}) \longrightarrow H^0(\mathcal{D})$  is an equivalence. The category  $\mathbf{DGcat}$  admits a cofibrantly generated Quillen model category structure whose weak equivalences are quasiisomorphisms [60]. Let us denote the homotopy category with respect to this model structure  $\mathbf{Hqe}$ . Being quasiequivalent is stronger than being Morita equivalent. Therefore, the category of DG categories up to quasiequivalence is bigger (has more non isomorphic objects) than  $\mathbf{NCS}$ . There is a canonical localisation functor  $\mathbf{Hqe} \longrightarrow \mathbf{NCS}$  inverting the Morita equivalences which are not quasiequivalences, which admits a section functor  $\mathcal{A} \longrightarrow \mathrm{per}_{dg}(\mathcal{A})$ , *i.e.*, a right adjoint to the canonical localisation functor.

Let us explain the construction of  $\mathrm{per}_{dg}(\mathcal{A})$  briefly. For a DG category  $\mathcal{A}$  a right  $\mathcal{A}$ -module, *i.e.*, a DG functor from  $\mathcal{A}^{\mathrm{op}}$  to the DG category of complexes over  $k$  is called *semifree* if it admits a countable filtration such that the subquotients are free DG modules (up to shifts), *i.e.*, modules formed by arbitrary sums of copies of  $\mathrm{Hom}(-, X)$  for some  $X \in \mathrm{Obj}(\mathcal{A})$ , possibly with shifts. Let us denote the category of semifree modules over  $\mathcal{A}$  by  $\mathrm{SF}(\mathcal{A})$ . The inclusion functor  $\mathrm{SF}(\mathcal{A}) \rightarrow \mathcal{A}^{\mathrm{op}}$ -modules induces an equivalence of triangulated categories between  $H^0(\mathrm{SF}(\mathcal{A}))$  and the derived category of  $\mathcal{A}$  [23]. The category  $\mathrm{per}_{dg}(\mathcal{A})$  is defined as the full DG subcategory of  $\mathrm{SF}(\mathcal{A})$  consisting of objects which become isomorphic to an object in  $\mathrm{per}(\mathcal{A})$  after passing on to the zeroth cohomology category. Roughly speaking,  $\mathrm{per}_{dg}(\mathcal{A})$  is a DG version of  $\mathrm{per}(\mathcal{A})$ , *i.e.*,  $H^0(\mathrm{per}_{dg}(\mathcal{A})) = \mathrm{per}(\mathcal{A})$ .

In fact, the category  $\mathbf{NCS}$  is equivalent to the full subcategory of  $\mathbf{Hqe}$  consisting of the pretriangulated (or Morita fibrant) DG categories.

### 3. ON NONCOMMUTATIVE MOTIVES

We begin by reviewing the classical notion of pure motives corresponding to smooth and projective varieties.

**3.1. Pure motives at a glance.** The main steps involved in the construction of effective pure motives from  $\mathbf{Var}$  are linearisation, pseudo-abelianisation and finally inversion of the Lefschetz motive, extending the tensor structure of  $\mathbf{Var}$  given by the fibre product over  $k$ . Letting  $\sim$  stand for any adequate relation, *e.g.*, rational, algebraic, homological or numerical, we define  $A^i(X)$  to be the abelian group of algebraic cycles of codimension  $i$  in  $X$  modulo  $\sim$ . We define an additive tensor category of correspondences, denoted by  $\mathbf{Corr}_\sim$ , keeping as objects those of  $\mathbf{Var}$  and setting

$$\mathbf{Corr}_\sim(X, Y) = \bigoplus_j A^{\dim j}(X \times Y_j),$$

where each  $Y_j$  is an irreducible component of  $Y$ .

**Definition 3.1.** An additive category  $\mathcal{D}$  is called *pseudo-abelian* if for any projector (idempotent)  $p \in \text{Hom}(X, X)$ ,  $X \in \text{Obj}(\mathcal{D})$  there exists a kernel  $\ker p$ .

There is a canonical *pseudo-abelian completion*  $\overline{\mathcal{D}}$  of any additive category  $\mathcal{D}$ . The objects of  $\overline{\mathcal{D}}$  are pairs  $(X, p)$ , where  $X \in \text{Obj}(\mathcal{D})$  and  $p \in \text{Hom}_{\mathcal{D}}(X, X)$  is an arbitrary projector. Define Hom sets as

$$\text{Hom}_{\overline{\mathcal{D}}}((X, p), (Y, q)) = \frac{\{f \in \text{Hom}_{\mathcal{D}}(X, Y) \text{ such that } fp = qf\}}{\{\text{subgroup of } f \text{ such that } fp = qf = 0\}}$$

We can apply this machinery to construct the pseudo-abelianisation of  $\mathbf{Corr}_\sim$ . In the resulting category the motive of  $\mathbb{P}^n$  decomposes as  $\mathbb{P}^n = \text{pt} \oplus \mathbb{L} \oplus \mathbb{L}^{\otimes 2} \oplus \cdots \oplus \mathbb{L}^{\otimes n}$ . The object  $\mathbb{L}$  is called the *Lefschetz motive* and it should be formally inverted in order to obtain the category of pure motives and morphisms should also be defined appropriately, but we gloss over these details here.

Restricting oneself to the subcategory of  $\mathbf{Var}$  consisting of connected curves and applying the above-mentioned three steps one obtains the category of motives of curves. This category admits a better description when  $\sim$  is chosen to be the rational equivalence relation and morphisms are tensored with  $\mathbb{Q}$ .

**Proposition 3.2** ([48]). *The category of motives of curves is equivalent to the category of abelian varieties up to isogeny.*

**Remark 3.3.** *The functor associates to a curve its Jacobian variety. It turns out that the category of abelian varieties up to isogeny is abelian and semisimple.*

The category of motives is expected to be semisimple and Tannakian (Jannsen showed that the category of motives modulo numerical equivalence is semisimple [30]). The category  $\mathbf{NCS}$  has some *motivic features*: it also has a tensor structure and an inner  $\mathcal{H}\text{om}$  functor. But not all objects  $T$  are *rigid*, *i.e.*, the canonical morphism  $T \otimes T^\vee \rightarrow \text{Hom}(T, T)$  is not an isomorphism for all  $T \in \mathbf{NCS}$ . However, the smooth and proper noncommutative spaces are rigid in the above sense.

**3.2. Towards noncommutative motives.** The first step of the construction of pure motives entails a linearisation of the category  $\mathbf{Var}$  by including correspondences. We have argued that correspondences induce DG functors (indeed, the kernel of a Fourier–Mukai transform should be thought of as a correspondence). The following Theorem [64] says that all DG-functors are described by a Fourier–Mukai *kernel*, and hence, more relevant to geometry than arbitrary exact functors between triangulated categories.

**Theorem 3.4** (Toën). *Let  $k$  be any commutative ring and let  $X$  and  $Y$  be quasicompact and separated schemes over  $k$  such that  $X$  is flat over  $\mathrm{Spec} k$ . Then there is a canonical isomorphism in  $\mathbf{NCS}$*

$$\mathcal{C}_{dg}(X \times_k Y) \xrightarrow{\sim} \mathcal{H}\mathrm{om}_c(\mathcal{C}_{dg}(X), \mathcal{C}_{dg}(Y)),$$

where  $\mathcal{H}\mathrm{om}_c$  denotes the full subcategory of  $\mathcal{H}\mathrm{om}$  formed by coproduct preserving quasi-functors, i.e., functors between the corresponding zeroth cohomology categories. Moreover, if  $X$  and  $Y$  are smooth and projective over  $\mathrm{Spec} k$ , we have a canonical isomorphism in  $\mathbf{NCS}$

$$\mathbf{Perf}_{dg}(X \times_k Y) \xrightarrow{\sim} \mathcal{H}\mathrm{om}(\mathbf{Perf}_{dg}(X), \mathbf{Perf}_{dg}(Y)),$$

where  $\mathbf{Perf}_{dg}$  denotes the full subcategory of  $\mathcal{C}_{dg}$ , whose objects are perfect complexes.

The above Theorem admits a natural generalisation to abstract DG categories (not necessarily of the form  $\mathcal{C}_{dg}(X)$  for some scheme  $X$ ), which can also be found in *ibid.*. The above theorem asserts an equivalence of categories. It can be suitably *decategorified*, in order to have an understanding of the morphisms on the right hand side.

For DG categories  $\mathcal{A}, \mathcal{B}$ , let us define  $\mathrm{rep}(\mathcal{A}, \mathcal{B})$  to be the full subcategory of the derived category  $D(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$  of  $\mathcal{A} - \mathcal{B}$ -bimodules formed by those  $M$ , which (under  $- \otimes_{\mathcal{A}} M : D(\mathcal{A}) \rightarrow D(\mathcal{B})$ ) send a representable  $\mathcal{A}$ -module to a  $\mathcal{B}$ -module, which, in  $D(\mathcal{B})$ , is isomorphic to a representable  $\mathcal{B}$ -module. The decategorified statement is that  $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$  in  $\mathbf{NCS}$  is canonically in bijection with the isomorphism classes of objects in  $\mathrm{rep}(\mathcal{A}, \mathcal{B})$  *ibid.*. If  $\mathcal{B}$  is pretriangulated, the objects of  $\mathrm{rep}(\mathcal{A}, \mathcal{B})$  are called *quasifunctors* as they induce honest functors  $H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$ .

Generalising this intuition we conclude that the morphisms in  $\mathbf{NCS}$  already contain all correspondences. However,  $\mathbf{NCS}$  is not an additive category as there is no abelian group structure on the set of morphisms. However, there is a semiadditive structure on  $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$  given by the direct sum of the kernels of two DG functors or objects in  $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ . We linearise them by passing on to the  $K_0$ -groups of the inner  $\mathcal{H}\mathrm{om}$  objects (see, for instance, [39],[59]).

It is also possible to talk about exact sequences in  $\mathbf{NCS}$ . We provide one formulation of an exact sequence of DG categories (see, e.g., Theorem 4.11 of [37] for other equivalent definitions).

**Definition 3.5.** *A sequence of DG categories*

$$\mathcal{A} \xrightarrow{P} \mathcal{B} \xrightarrow{I} \mathcal{C}$$

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such that  $IP = 0$  is called exact if and only if  $P$  induces an equivalence of  $\text{per}(\mathcal{A})$  onto a thick subcategory of  $\text{per}(\mathcal{B})$  and  $I$  induces an equivalence between the idempotent closure of the Verdier quotient  $\text{per}(\mathcal{B})/\text{per}(\mathcal{A})$  and  $\text{per}(\mathcal{C})$ .

**Remark 3.6.** In the classical setting, if  $X$  is a quasicompact quasiseparated scheme,  $U \subset X$  a quasicompact open subscheme and  $Z = X \setminus U$ , then the following sequence

$$\mathbf{Perf}_{dg}(X)_Z \longrightarrow \mathbf{Perf}_{dg}(X) \longrightarrow \mathbf{Perf}_{dg}(U)$$

is exact according to the definition, where  $\mathbf{Perf}_{dg}(X)_Z$  denotes the full subcategory of  $\mathbf{Perf}_{dg}(X)$  of perfect complexes supported on  $Z$ .

One knows that there is a well-defined  $K$ -theory functor on  $\mathbf{NCS}$ , which agrees with Quillen's  $K$ -theory of an exact category  $\mathcal{B}$ , when applied to the Drinfeld quotient of  $\mathcal{C}_{dg}^b(\mathcal{B})$  by its subcategory of acyclic complexes. Now we define a noncommutative analogue of the category of correspondences (a naïve version). A more sophisticated approach should treat the category enriched over *spectra*, a construction of which can be found in [57].

**Definition 3.7.** The category of noncommutative correspondences  $\mathbf{NCC}$  is the category defined as:

- $\text{Obj}(\mathbf{NCC}) = \text{Obj}(\mathbf{NCS})$
- $\text{Hom}_{\mathbf{NCC}}(\mathcal{C}, \mathcal{D}) = K_0(\text{rep}(\mathcal{C}, \mathcal{D}))$

As a motivation we mention two Theorems: the first Theorem ensures *linearisation* of  $\mathbf{NCS}$ , while the second one shows compatibility with localisation.

**Theorem 3.8.** [23, 24, 59] A functor  $F$  from  $\mathbf{NCS}$  to an additive category factors through  $\mathbf{NCC}$  if and only if for every exact DG category  $\mathcal{B}$  endowed with two full exact DG subcategories  $\mathcal{A}, \mathcal{C}$  which give rise to a semiorthogonal decomposition  $H^0(\mathcal{B}) = (H^0(\mathcal{A}), H^0(\mathcal{C}))$  in the sense of [13], the inclusions induce an isomorphism  $F(\mathcal{A}) \oplus F(\mathcal{C}) \xrightarrow{\sim} F(\mathcal{B})$ .

Such a functor is called an *additive invariant* of noncommutative spaces. The simplest example is  $\mathcal{A} \mapsto K_0(\text{per}(\mathcal{A}))$ .

**Theorem 3.9.** [24] The functor  $\mathcal{A} \mapsto K(\mathcal{A})$  (Waldhausen  $K$ -theory) yields, for each short exact sequence  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  in  $\mathbf{NCS}$ , a long exact sequence

$$\cdots \longrightarrow K_i(\mathcal{A}) \longrightarrow K_i(\mathcal{B}) \longrightarrow K_i(\mathcal{C}) \longrightarrow \cdots \longrightarrow K_0(\mathcal{B}) \longrightarrow K_0(\mathcal{C}).$$

**Remark 3.10.** The category  $\mathbf{NCC}$  is additive and the composition is induced by that of  $\mathbf{NCS}$ . Certain non-isomorphic objects of  $\mathbf{NCS}$  become isomorphic in  $\mathbf{NCC}$ , e.g., it is shown in [35] that each finite dimensional algebra of finite global dimension becomes isomorphic to a product of copies of  $k$  in  $\mathbf{NCC}$ , whereas such a thing is true in  $\mathbf{NCS}$  if and only if the algebra is semisimple.

We should perform a formal idempotent completion (or pseudo-abelian completion) of  $\mathbf{NCC}$  as discussed in Subsection 3.1 in order to obtain the category of noncommutative motives, which is denoted by  $\mathbf{NCM}$ . It follows from Beilinson's description of the derived category of coherent sheaves on  $\mathbb{P}^n$  [3] and the above remark that  $\mathcal{C}_{dg}(\mathbb{P}^1) \simeq \mathcal{C}_{dg}(\mathbb{A}^1) \oplus \mathcal{C}_{dg}(\text{pt})$  is also isomorphic in  $\mathbf{NCC}$  to  $\mathcal{C}_{dg}(\text{pt}) \oplus \mathcal{C}_{dg}(\text{pt})$ , whence  $\mathcal{C}_{dg}(\mathbb{A}^1) \simeq \mathcal{C}_{dg}(\text{pt})$ , i.e., the Lefschetz motive is isomorphic to the identity element.

A careful reader should have noticed that we have glossed over the issue of the choice of the equivalence relation, which was central to the construction of the category of pure motives in the classical setting. Manin mentioned in [48] (end of Section 3) that every cohomology theory should be a cohomological functor on the category of  $\text{Corr}_\sim$ , *i.e.*, every correspondence in  $\text{Corr}_\sim(X, Y)$  should induce a well-defined morphism  $H^*(X) \rightarrow H^*(Y)$ . Now we turn the argument around. Elements of  $\text{rep}(\mathcal{C}, \mathcal{D})$  induce morphisms between  $\mathcal{C}$  and  $\mathcal{D}$ . Our spaces are defined in terms of the (quasicoherent) cohomologies that they admit. Mostly cohomology theories appear as cohomology groups of a certain canonically defined cochain complex satisfying a bunch of axioms. We pretend that a morphism (a functor) in NCM is a morphism between the cohomology theories on the two spaces, as if given by some correspondence. If the question about universal cohomology theory is resolved, then probably one would like to argue that the elements of  $\text{rep}(\mathcal{C}, \mathcal{D})$  are the ones which induce distinct morphisms between their universal cohomologies. If that turns out to be false then one can call an equivalence relation *universal* if it identifies two correspondences which induce isomorphic morphisms between the corresponding *universal cohomology theories* and then consider correspondences modulo this equivalence relation. Note that in NCM we set the Grothendieck group of  $\text{rep}(\mathcal{A}, \mathcal{B})$  as morphisms between  $\mathcal{A}$  and  $\mathcal{B}$ . Chow correspondences are obtained by taking the rational equivalence relation. The connection should be an analogue of the *Chern character* map which identifies the K-theory with the Chow group after tensoring with  $\mathbb{Q}$ . The readers are referred to [14] for a possibly relevant treatment of the Chern character.

**3.3. Motivic measures and motivic zeta functions.** We present a rather simplistic point of view on motivic measures. With respect to a motivic measure it is possible to develop a theory of *motivic integration* (*see, e.g.*, [20]), which we shall not discuss here. This technology was invented by Kontsevich drawing inspiration from the works of Batyrev. A useful and instructive reference is, *e.g.*, [44].

Let  $\text{Sch}_k$  be the category of reduced schemes of finite type (or reduced varieties) over  $k$ . Consider the Grothendieck ring of  $\text{Sch}_k$ , denoted by  $K_0(\text{Sch}_k)$ , which is defined as the free abelian group generated by isomorphism classes of objects in  $\text{Sch}_k$  modulo relations (often called scissor-congruence relations)

$$(2) \quad [X] = [Z] + [X \setminus Z],$$

where  $Z$  is a closed subscheme of  $X$ . The multiplication is given by the fibre product over  $k$ . There is a unit given by the class of  $\text{Spec } k$ .

Every  $k$ -variety admits a finite stratification  $X = X^0 \supset X^1 \supset \dots \supset X^{d+1} = \emptyset$  such that  $X^k \setminus X^{k+1}$  is smooth. Moreover, any two such stratifications admit a common refinement. Therefore  $[X] = \sum_k [X^k \setminus X^{k+1}]$  is unambiguously defined and, in fact, it can be shown that  $K_0(\text{Sch}_k)$  is generated by complete and nonsingular varieties. The structure of  $K_0(\text{Sch}_k)$  as a ring is rather complicated. It is known that it is not an integral domain [54]. However, it admits interesting ring homomorphisms to some rings, which turn out to be quite useful in various cases.

Let  $A$  be any commutative ring. An  $A$ -valued *motivic measure* is a ring homomorphism  $\mu_A : K_0(\text{Sch}_k) \rightarrow A$ . We write  $\mu = \mu_A$  if there is no chance of confusion. If  $A$  has a unit the homomorphism is required to be unital.

**Example 2.** Let  $k = \mathbb{C}$ ,  $A = \mathbb{Z}$  and  $\mu(X) = \chi_c(X)$ , i.e., the Euler characteristic with compact supports.

**Example 3.** Let  $k = \mathbb{C}$ ,  $A = K_0(\text{HS})$ , i.e., the Grothendieck ring of Hodge structures and  $\mu(X) = \chi_h(X)$  such that

$$\chi_h(X) = \sum_r (-1)^r [\text{H}_c^r(X, \mathbb{Q})] \in K_0(\text{HS}),$$

which is called the Hodge characteristic of  $X$ .

**Example 4.** Let  $k = \mathbb{F}_q$ ,  $A = \mathbb{Z}$  and  $\mu(X) = \#X(\mathbb{F}_q)$ , i.e., the number of  $\mathbb{F}_q$ -points.

Let us fix an  $A$ -valued motivic measure  $\mu$  and, for a smooth  $X \in \text{Sch}_k$ , let  $X^{(n)}$  denote the  $n$ -fold symmetric product of  $X$ . Set  $X^{(0)} := \text{Spec } k$ . Then associated to  $\mu$  there is a *motivic zeta function* (possibly due to Kapranov [31]) of  $X$  defined by the formal series

$$(3) \quad \zeta_\mu(X, t) = \sum_{n=0}^{\infty} \mu(X^{(n)}) t^n \in A[[t]].$$

**Example 5.** If  $k = \mathbb{F}_q$ ,  $A = \mathbb{Z}$  and  $\mu(X) = \#X(\mathbb{F}_q)$  as in Example 4 one recovers the usual Hasse–Weil zeta function of  $X$ . Indeed, the  $\mathbb{F}_q$ -valued points of  $X^{(n)}$  correspond to the effective divisors of degree  $n$  in  $X$ .

Let us denote  $\mu(\mathbb{A}_k^1)$  by  $\mathbb{L}$ . Then we have the following rationality statement for curves (see Theorem 1.1.9 *ibid.*).

**Theorem 3.11.** *If  $X$  is any one dimensional variety (not necessarily non-singular) of genus  $g$ , then  $\zeta_\mu(X, t)$  is rational. Furthermore, the rational function  $\zeta_\mu(X, t)(1-t)(1-\mathbb{L}t)$  is actually a polynomial of degree  $\leq 2g$  and satisfies the functional equation below.*

$$(4) \quad \zeta_\mu(X, 1/\mathbb{L}t) = \mathbb{L}^{1-g} t^{2-2g} \zeta_\mu(X, t)$$

**Remark 3.12.** The rationality statement fails to be true in higher dimensions, e.g., if  $X$  is a complex projective non-singular surface of genus  $\geq 2$  [41]. In fact, a complex surface  $X$  has a rational motivic zeta function if and only if it has Kodaira dimension  $-\infty$  [42].

**3.4. Noncommutative Calabi–Yau spaces.** This section attempts to introduce zeta functions of noncommutative curves *in a motivic framework* and possibly extract some arithmetic information out of them. That the zeta functions of varieties contain crucial arithmetic information is a gospel truth by now.

Before we move forward let us mention that such ideas are prevalent in noncommutative geometry, e.g., Connes’ spectral realisation of the zeros of the Riemann zeta function [17, 16]. Some other important works in this direction are [18], [21, 22], [52] and [29], to mention only a few. Also the readers should take a look at [49] for a more holistic point of view.

Following Proposition 3.2 we argue that the category of noncommutative motives of non-commutative curves should be equivalent to the full subcategory of  $\text{NCM}$  generated by DG categories which resemble those of abelian varieties, i.e., the inclusion of abelian varieties inside  $\text{NCM}$  (see Equation (2.3)). Given an abelian surface the cokernel of the multiplication by 2 map (isogeny) is a Kummer surface with 16 singular points, whose (minimal) resolution

of singularities is a  $K3$  surface. It is an example of a Calabi–Yau manifold of dimension 2. So even if we look at motives of curves Calabi–Yau varieties show up rather naturally. We propose to treat such varieties as they are, rather than working up to isogenies. Calabi–Yau varieties are interesting from the point of view of physics as well. We assume that a Calabi–Yau variety is just a variety, whose canonical class is trivial (no assumption on the fundamental group).

In a  $k$ -linear category  $\mathcal{A}$  an additive autoequivalence  $S$  is called a Serre functor if there exists a bifunctorial isomorphism  $\text{Hom}(A, B) \xrightarrow{\sim} \text{Hom}(B, SA)^*$  for any two  $A, B \in \text{Obj}(\mathcal{A})$ . If it exists it is unique up to isomorphism. If  $X$  is a smooth projective variety of dimension  $n$ , the Serre functor is given by  $(-\otimes\omega_X)[n]$ , where  $\omega_X$  is the canonical sheaf of  $X$ . The existence of a Serre functor corresponds to that of Grothendieck–Serre duality.

**Definition 3.13.** *A DG category  $\mathcal{C}$  in NCS is called a noncommutative Calabi–Yau space of dimension  $n$  if  $H^0(\mathcal{C})$  is triangulated (i.e.,  $\mathcal{C}$  is pretriangulated as in Definition 2.4) with the finiteness condition  $\sum_p \dim_k \text{Hom}_{H^0(\mathcal{C})}(X, Y[p]) < \infty$  for all  $X, Y \in \text{Obj}(H^0(\mathcal{C}))$ , and if there exists a natural isomorphism between the Serre functor and  $[n]$ . In other words, there exists bifunctorial isomorphisms  $\text{Hom}(A, B) \xrightarrow{\sim} \text{Hom}(B, A[n])^*$  in  $H^0(\mathcal{C})$ .*

Kontsevich originally defined a noncommutative Calabi–Yau space as a small triangulated category satisfying the strong finiteness condition mentioned above, with an isomorphism between the Serre functor and  $[n]$ . We have enhanced it to the DG level. It should be borne in mind that the homotopy category of a pretriangulated category is idempotent complete. It is clear that if  $X$  is a Calabi–Yau variety then  $\mathcal{C}_{dg}(X)$  is a Calabi–Yau space in the above sense. Purely at the triangulated level there are other interesting examples of Calabi–Yau spaces of dimension 2 arising from quiver representations and commutative algebra cf. Section 4 of [38]. When such a triangulated Calabi–Yau category of dimension  $d$  is endowed with a *cluster tilting subcategory* it is possible to construct a Calabi–Yau DG category (in the above sense) of dimension  $d + 1$  [61].

Let us denote by  $\text{NCM}_{CY}$  the full additive subcategory of  $\text{NCM}$  consisting of noncommutative Calabi–Yau spaces.

**Example 6.** *It is expected that via a noncommutative version of the construction of the Jacobian of a curve the category of motives of noncommutative curves can be seen as a full subcategory of  $\text{NCM}_{CY}$ . The way to view an abelian variety in this setting is not clear to the author yet. The category  $\text{NCM}_{CY}$  contains honest elliptic curves (as they are their own Jacobians) as given by the inclusion of classical geometry in this setting (see Equation (2.3)). The noncommutative torus  $\mathbb{T}_\theta^\tau$  is also included via its DG derived category of holomorphic bundles. It is isomorphic to  $\mathcal{C}_{dg}(X_\tau)$ , where  $X_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ , via a Fourier–Mukai type functor (see Proposition 3.1 [53]). J. Block [7, 8] suggests a more conceptual framework for such dualities to exist. The rough idea is to construct a differential graded algebra from a complex torus  $X$  associated to a deformation parameter in  $\text{HH}^2(X)$  and look at the DG category  $DG(X)$  of twisted complexes, i.e., DG modules over that algebra equipped with a super connection compatible with the differential of the algebra. One can construct a (curved) differential graded algebra corresponding to the dual torus as well with a curvature contribution given by the deformation parameter, whose DG category of twisted complexes will be quasiequivalent to  $DG(X)$  via some sort of a deformed Poincaré bundle (essentially a correspondence).*

### 3.5. The motivic ring of NCS.<sup>4</sup>

Let us recall from Section 3.3 that an  $A$ -valued motivic measure  $\mu$  is a ring homomorphism from  $K_0(\mathbf{Sch}_k) \rightarrow A$ . We have replaced the category of  $k$ -schemes by the category of noncommutative spaces NCS. We need an appropriate notion of the Grothendieck ring of NCS, which we would like to call the motivic ring of NCS.

Since every object in NCS is quasiequivalent to a pretriangulated DG category we seek a Grothendieck ring of pretriangulated DG categories. In [12] the authors precisely construct a Grothendieck ring of pretriangulated DG categories, which is essentially the Grothendieck ring of  $\mathbf{Hqe}$ . It was pointed out by the authors that it is crucial to work with DG categories (and not honest triangulated ones) as the tensor product of two triangulated categories does not have a natural triangulated structure in general. Let us briefly recall their construction.

The Grothendieck ring  $\mathcal{G}$  is generated as a free abelian group by the isomorphism classes of pretriangulated DG categories in NCS (or quasiequivalence classes of objects in  $\mathbf{DGcat}$ ) modulo relations analogous to those of  $K_0(\mathbf{Sch}_k)$ . The authors reinterpret the excision relations as those coming from *semiorthogonal decompositions* (see [13] for the details of semiorthogonal decomposition). One writes  $[\mathcal{B}] = [\mathcal{A}] + [\mathcal{C}]$  if and only if there exist representatives  $\mathcal{A}', \mathcal{B}', \mathcal{C}'$  in  $[\mathcal{A}], [\mathcal{B}], [\mathcal{C}]$  respectively such that

- (1)  $\mathcal{A}', \mathcal{C}'$  are DG subcategories of  $\mathcal{B}'$ ,
- (2)  $H^0(\mathcal{A}'), H^0(\mathcal{C}')$  are admissible subcategories of  $H^0(\mathcal{B}')$ ,
- (3)  $(H^0(\mathcal{A}'), H^0(\mathcal{C}'))$  is a semiorthogonal decomposition of  $H^0(\mathcal{B}')$ .

**Remark 3.14.** Part (3) implies that  $H^0(\mathcal{A}') = (H^0(\mathcal{C}'))^\perp$ , which is Lemma 2.25 in [12]. An exact sequence  $\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}$  of pretriangulated DG categories (cf. Definition 3.5) induces an exact sequence of honest triangulated categories  $H^0(\mathcal{A}) \longrightarrow H^0(\mathcal{B}) \longrightarrow H^0(\mathcal{C})$  by definition. However, existence of a semiorthogonal decomposition is a stronger condition. It says that  $H^0(\mathcal{C})$  is a triangulated subcategory of  $H^0(\mathcal{B})$  and  $H^0(\mathcal{A}) = (H^0(\mathcal{C}))^\perp$ , i.e., the sequence is split (cf. Theorem 3.8). It is plausible that one obtains something sensible by allowing all possible exact sequences as relations.

The product  $\bullet$  is defined as follows:

$$\mathcal{A}_1 \bullet \mathcal{A}_2 := \text{per}_{dg}(\mathcal{A}_1 \otimes \mathcal{A}_2),$$

where  $\text{per}_{dg}(\mathcal{A})$  is a pretriangulated DG category described in subsection 2.4.

The product  $\bullet$  preserves quasiequivalences of DG categories and hence descends to a product on  $\mathcal{G}$ . It is proven in [12] that the product is associative and commutative. There is a unit given by the class of  $\mathcal{C}_{dg}^b(k)$ , i.e., the DG category of finite dimensional chain complexes over  $k$ . That this product corroborates the fibre product of varieties is justified by Theorem 6.6 *ibid.*

**Remark 3.15.** The motivic ring of NCS should be the above-mentioned ring with quasiequivalences replaced by Morita equivalence. There will be a canonical ring homomorphisms corresponding to the localisation functor  $\mathbf{Hqe} \longrightarrow \mathbf{NCS}$ . Since a quasiequivalence is also a Morita

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<sup>4</sup>The author was kindly notified by B. Keller and the referee that the motivic ring constructed in [12], which is what we describe here, is actually isomorphic to the 0 ring [59]. However, with some appropriate finiteness conditions thrown in, this problem has been rectified, (e.g., [58]).

equivalence the ring homomorphism will be surjective identifying elements which are Morita equivalent but not quasiequivalent.

It follows that the product of two noncommutative Calabi–Yau categories is again a noncommutative Calabi–Yau category, *i.e.*,  $\mathcal{A} \bullet \mathcal{B}$  is a noncommutative Calabi–Yau DG category of dimension  $m + n$  for  $\mathcal{A}, \mathcal{B} \in \text{NCM}_{\text{CY}}$  of dimensions  $m, n$  respectively. Indeed, the finiteness condition follows from Künneth formula. To check the existence of the Serre functor first observe that

$$\begin{aligned}
& \text{Hom}_{H^0(\mathcal{A} \otimes \mathcal{B})}(A \otimes B, A' \otimes B'[m+n]) \\
= & H^0 \text{Hom}_{\mathcal{A} \otimes \mathcal{B}}(A, A') \otimes \text{Hom}_{\mathcal{A} \otimes \mathcal{B}}(B, B')[m+n] \\
= & H_{\mathcal{A} \otimes \mathcal{B}}^{m+n}(\text{Hom}(A, A') \otimes \text{Hom}(B, B')) \\
= & H^{m+n}((\oplus_i \text{Hom}_{\mathcal{A} \otimes \mathcal{B}}^i(A, A') \otimes \text{Hom}_{\mathcal{A} \otimes \mathcal{B}}^{k-i}(B, B'))^\bullet) \\
= & \oplus_l \text{Hom}_{H^0(\mathcal{A} \otimes \mathcal{B})}^{l+m}(A, A') \otimes \text{Hom}_{H^0(\mathcal{A} \otimes \mathcal{B})}^{n-l}(B, B') \\
= & \oplus_l \text{Hom}_{H^0(\mathcal{A} \otimes \mathcal{B})}^l(A, A'[m]) \otimes \text{Hom}_{H^0(\mathcal{A} \otimes \mathcal{B})}^{-l}(B, B'[n]) \\
= & \oplus_l \text{Hom}_{H^0(\mathcal{A} \otimes \mathcal{B})}^l(A', A)^* \otimes \text{Hom}_{H^0(\mathcal{A} \otimes \mathcal{B})}^{-l}(B', B)^* \\
= & \left( \oplus_l \text{Hom}_{H^0(\mathcal{A} \otimes \mathcal{B})}^l(A', A) \otimes \text{Hom}_{H^0(\mathcal{A} \otimes \mathcal{B})}^{-l}(B', B) \right)^* \\
= & H^0(\text{Hom}_{\mathcal{A} \otimes \mathcal{B}}^\bullet(A' \otimes B', A \otimes B))^* \\
= & \text{Hom}_{H^0(\mathcal{A} \otimes \mathcal{B})} \text{Hom}(A' \otimes B', A \otimes B)^*
\end{aligned}$$

This proves that  $H^0(\mathcal{A} \otimes \mathcal{B})$  has the right Serre functor  $[n+m]$ .

Now, it follows from [47] that the existence of the Serre functor  $[n]$  is equivalent to the isomorphism  $\text{Hom}_k(\mathcal{A}(-, ?), k) \simeq \mathcal{A}(?, -[n])$  in  $D(\mathcal{A} \otimes \mathcal{A}^{op})$ . Since  $\mathcal{A} \mapsto \text{per}_{dg}(\mathcal{A})$  is a monoidal Morita isomorphism this equivalence of bimodules is preserved under the functor.

A  $\mathcal{G}$ -valued motivic measure (the ring homomorphism being the identity map) is a universal one. Let us denote the class of  $X \in \text{NCS}$  inside  $\mathcal{G}$  by  $[X]$ . Given any variety  $Y$  we know that  $\mathcal{C}_{dg}(Y \times Y)$  is quasiequivalent (in particular, Morita equivalent) to  $\mathcal{H}om_c(\mathcal{C}_{dg}(Y), \mathcal{C}_{dg}(Y))$ . Let us define inductively  $X^n = \mathcal{H}om_c(X^{n-1}, X)$  and  $X^1 = X$ . Then the universal  $\mathcal{G}$ -valued motivic zeta function of  $X \in \text{NCS}$  is given by

$$\zeta_{\mu_{\mathcal{G}}}(X, t) = 1 + \sum_{n=1}^{\infty} [X^n] t^n \quad \in \mathcal{G}[[t]].$$

It is shown in [12] that there is a canonical surjective ring homomorphism  $K_0(\mathbf{Sch}_k) \rightarrow \mathcal{G}_{hon}$  with  $(\mathbb{L} - 1)$  in the kernel, where  $\mathcal{G}_{hon}$  is the subring of  $\mathcal{G}$  generated by certain pretriangulated DG categories associated to honest smooth projective varieties over  $k$ .

**Remark 3.16.** *Since the DG category of holomorphic bundles on the noncommutative torus  $\mathbb{T}_\theta^\tau$  is equivalent to  $\mathcal{C}_{dg}(X_\tau)$ , where  $X_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$  [53], the zeta functions of  $\mathbb{T}_\theta^\tau$  and  $X_\tau$  are the same. However, the equivalence is given by a non-trivial Fourier–Mukai type (correspondence) functor. The B-model of a conformal field theory associates to a complex torus its*

*derived category of coherent sheaves. The equivalence perhaps indicates that deforming the complex torus to a noncommutative torus does not produce anything new for the B-model.*

One nagging point is that certain natural topological constructions do not allow us to define a category in which the composition of morphisms obeys associativity (it is associative only up to homotopy). Hence some mathematicians have resorted to working with  $A_\infty$  categories which encode such properties, *e.g.*, [40]. The world of  $A_\infty$  categories subsumes that of DG categories. However, one knows that every  $A_\infty$  category is quasiisomorphic to a DG category in a functorial manner (*e.g.*, [19]).

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